Solving ODEs by using the Complementary Function and Particular Integral

An ordinary differential equation (ODE)\(^1\) is an equation that relates a summation of a function \(x(t)\) and its derivatives. In this document we consider a method for solving second order ordinary differential equations of the form

\[
\frac{d^2 x}{dt^2} + a \frac{dx}{dt} + bx = f(t),
\]

where \(a\) and \(b\) are constants, through deriving the \textit{complementary function} and \textit{particular integral}.

\textbf{Method}

Given an ordinary differential equation in \(x(t)\):

\[
\frac{d^2 x}{dt^2} + a \frac{dx}{dt} + bx = f(t),
\]

The solution is found through augmenting the results of two solution methods called the \textit{complementary function} and the \textit{particular integral}.

\begin{enumerate}
  \item \textbf{Complementary Function}

    The first step is to find the \textit{complementary function}, that is the general solution of the relevant \textit{homogeneous equation}

    \begin{enumerate}
      \item The homogeneous equation is derived by simply replacing the \(f(t)\) by zero:
      \[
      \frac{d^2 x}{dt^2} + a \frac{dx}{dt} + bx = 0.
      \]

      \item We presume that a solution of the homogeneous equation has a form \(x(t) = e^{mt}\), where \(m\) is either real or complex\(^2\), which is similar to the concept of using \textit{phasors}\(^3\) to solve differential equations. The upshot of this is that \(\frac{dx}{dt} = me^{mt}\) and \(\frac{d^2 x}{dt^2} = m^2 e^{mt}\) and the substitution of these terms into the homogeneous equation and cancelling out the common \(e^{mt}\) term gives the \textit{auxiliary equation}:
      \[
      \frac{d^2 x}{dt^2} + a \frac{dx}{dt} + bx = 0.
      \]

      \item The auxiliary equation is a quadratic equation which needs to be solved\(^4\) so that we can progress towards the complementary function. We consider three classes of outcomes and the associated complementary function in the following table.
    \end{enumerate}

\end{enumerate}

\footnotesize
\begin{flushleft}
\begin{tabular}{|c|c|c|}
\hline
\textbf{Ordinary Differential Equations} & \textbf{Complex Numbers} & \textbf{Phasors} \\
\hline
\textbf{Solving Quadratic Equations} & & \\
\hline
\end{tabular}
\end{flushleft}

\footnotesize
Auxiliary equation has... | Complementary function is...
---|---
two real roots \( m_1 \) and \( m_2 \) | \( x = A e^{m_1 t} + B e^{m_2 t} \)
one repeated real root \( m \) | \( x = (A + B t) e^{m t} \)
complex conjugate roots \( \alpha \pm i\beta \) | \( x = e^{\alpha t}(A \cos(\beta t) + B \sin(\beta t)) \)

2. **Particular Integral**

(a) Determine the general form of the particular integral. Having found the solution to the auxiliary equation, the next step is to re-introduce the function \( f(t) \). The solution to the equation based on the function \( f(t) \) is called the particular integral. The particular integral function is based on substituting a trial form of solution that is based on the function \( f(t) \). The following table shows typical functions \( f(t) \) and typical trial solutions. Note C’s, D’s denote constants.

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>Trial solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant</td>
<td>( C )</td>
</tr>
<tr>
<td>polynomial(^5) of degree ( n )</td>
<td>( C_n t^n + C_{n-1} t^{n-1} + \cdots + C_1 t + C_0 )</td>
</tr>
<tr>
<td>a multiple of ( e^{ct} )</td>
<td>( C e^{ct} )</td>
</tr>
<tr>
<td>multiple(s) of ( \sin(\omega t) ) and ( \cos(\omega t) )</td>
<td>( C \sin(\omega t) + D \cos(\omega t) )</td>
</tr>
</tbody>
</table>

The trial functions can be memorised since they are similar to the given function \( f(t) \) or the typical forms of function that we would expect through repeated differentiation of \( f(t) \). Note that if \( f(t) \) is a combination of the above then the trial solution should be made up of a similar combination.

(b) Determine the particular form of the particular integral. The general form of the particular integral is substituted back into the differential equation and the resulting solution is called the particular integral.

3. **General Solution**

Determine the general solution to the differential equation. The general solution is the sum of the complementary function and the particular integral.

4. **Particular Solution**

The unknown coefficients in the general solution are found by imposing the boundary conditions on the general solution.

---

\(^5\) Polynomials
Example

Find $x$ when $\frac{d^2 x}{dt^2} - 7\frac{dx}{dt} + 12x = 2$ and $x = 1$ and $\frac{dx}{dt} = 5$ when $t = 0$.

Answer:

1. Determine the complementary function

   (a) The homogeneous equation is
   
   $$\frac{d^2 x}{dt^2} - 7\frac{dx}{dt} + 12x = 0.$$  

   (b) Hence the auxiliary equation is
   
   $$m^2 - 7m + 12 = 0.$$  

   (c) The auxiliary equation can be solved through factorisation:
   
   $$(m - 3)(m - 4) = 0$$  

   and hence the auxiliary equation has two real solutions $m = 3, 4$. Hence the complementary function is
   
   $$x_{CF}(t) = Ae^{3t} + Be^{4t}.$$  

2. Determine the particular integral

   Since $f(t) = 2$ then let us try $x_{PI}(t) = C$, following the table above. Since the derivatives of $x_{PI}(t)$ are zero then the substitution of this into the original ordinary differential equation gives
   
   $$12C = 2$$  

   and hence
   
   $$C = \frac{1}{6} \text{ and } x_{PI}(t) = \frac{1}{6}.$$  

3. The general solution is
   
   $$x(t) = x_{CF}(t) + x_{PI}(t) = Ae^{3t} + Be^{4t} + \frac{1}{6}.$$  

4. With the first initial conditions, when $t = 0$, $x = 1$.
   
   Hence $A + B + \frac{1}{6} = 1$ or $A + B = \frac{5}{6}$.

   With the second initial condition, when $t = 0$, $\frac{dx}{dt} = 5$.
   
   $\frac{dx}{dt} = 3Ae^{3t} + 4Be^{4t}$ and hence $3A + 4B = 5$.

   Solving the simultaneous equations in $A$ and $B$ gives $A = \frac{5}{3}B = \frac{5}{2}$ and hence the solution
   
   $$x(t) = x_{CF}(t) + x_{PI}(t) = -\frac{5}{3}e^{3t} + \frac{5}{2}e^{4t} + \frac{1}{6}.$$