Repeated Integration

A repeated integral involves integrating a function of two or more variables over surfaces (2 variables), volumes (3 variables), and so on. Generally this involves integration[1] for each variable in turn until a result is determined. For example repeated integration involves finding the solution of

$$\iint f(x,y)dxdy \quad \text{or} \quad \iiint f(x,y,z)dxdydz\,.$$

Separable Functions

The simplest case of repeated integration is in the case in which the integrand is separable.

For example if $f(x,y) = p(x)q(y)$ then $f(x,y)$ is said to be separable and we may write

$$\iint f(x,y)dxdy = \iint p(x)q(y)dxdy = \left(\int p(x)dx\right)\left(\int q(y)dy\right).$$

Example

Find $\int_0^{\pi/4}\int_2^3 \cos(x)e^y\,dydx$

Answer

$$\int_0^{\pi/4}\int_2^3 \cos(x)e^y\,dydx = \int_0^{\pi/4} \cos(x)\,dx \int_2^3 e^y\,dy$$

$$= [-\sin(x)]_0^{\pi/4}[e^y]_2^3 = \frac{1}{\sqrt{2}}(e^3 - e^2)\,.$$

Fixed Limits

It the limits are fixed then the domain of integration is rectangular, cubic etc. If the limits are fixed then we can directly change the order of integration:

$$\int_a^b\int_c^d f(x,y)dydx = \int_c^d\int_a^b f(x,y)dxdy\,.$$

Changing the order of integration does not change the result.

[1] Integration

Example

Find $\int_1^2 \int_2^3 (x^2 + 2y)dydx$

Answer

$$\int_1^2 \int_2^3 (x^2 + 2y)dydx = \int_1^2 [x^2y + y^2]_1^2 \, dx = \int_1^2 (x^2 + 3) \, dx$$
$$= \left[\frac{x^3}{3} + 3x\right]_1^2 = \left(\frac{8}{3} + 6\right) - \left(\frac{1}{3} + 3\right) = \frac{16}{3}.$$

Example

Changing the order of integration, find $\int_1^2 \int_2^3 (x^2 + 2y)dydx$

Answer

$$\int_1^2 \int_2^3 (x^2 + 2y)dydx = \int_2^3 \int_1^2 (x^2 + 2y)dxdy = \int_2^3 \left[\frac{x^3}{3} + 2yx\right]_1^2 dy$$
$$= \int_2^3 \left(\frac{7}{3} + 2y\right) dy = \left[\frac{7}{3}y + y^2\right]_1^2 = \left(\frac{14}{3} + 4\right) - \left(\frac{7}{3} + 1\right) = \frac{16}{3}.$$

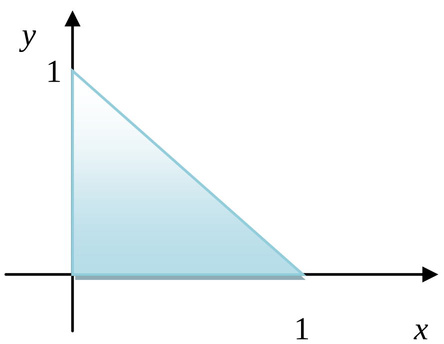
Note that the result is identical to the result in the previous example.

Variable Limits

For more complicated domains of integration it is necessary to have variable limits. In this case the limits of integration cannot be directly changed.

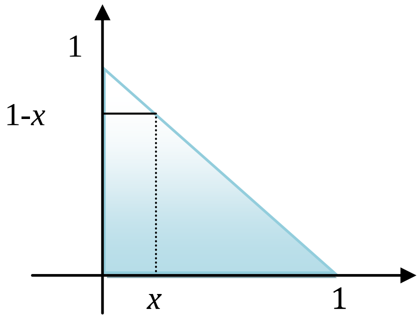
Example

Express the following triangular domain of integration in both orders.

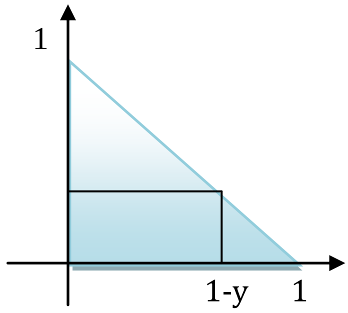


Answer

If we consider the x-direction in the first instance then we can see that x ranges from 0 to 1. If we now consider the y-direction, we can see that for any particular value of x, y ranges from 0 to (1-x).



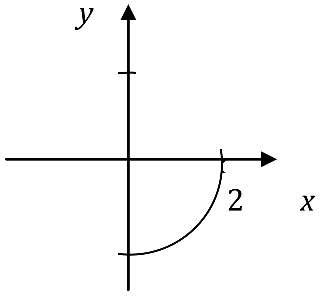
Hence $\iint_{\Delta} f(x,y)dydx = \int_0^1 \int_0^{1-x} f(x,y)dydx$ .If we consider the y-direction in the first instance then we can see that x ranges from 0 to 1. If we now consider the y-direction, we can see that for any particular value of y, x ranges from 0 to (1-y).



Hence $\iint_{\Delta} f(x,y)dydx = \int_0^1 \int_0^{1-y} f(x,y)dxdy$.

Example

Find $\int_D xydy\,dx$ where D is the quarter circle illustrated in the following diagram.



Change the order of integration and show that the same result is obtained.

Answer

As x varies from 0 to 2, y varies from $-\sqrt{4-x^2}$ to 0.

Hence $\int_D xy\,dydx = \int_0^2 \int_{-\sqrt{4-x^2}}^0 xy\,dydx = \frac{1}{2}\int_0^2 [xy^2]_{-\sqrt{4-x^2}}^0\ dx$

$= -\frac{1}{2}\int_0^2 x(4-x^2)\ dx = \frac{1}{2}\int_0^2 (x^3 - 4x)\ dx = \frac{1}{2}\left[\frac{x^4}{4} - 2x^2\right]_0^2 = -2.$

Alternatively, as y varies from -2 to 0, x varies from 0 to $\sqrt{4-y^2}$,

Hence $\int_D xy\,dydx = \int_{-2}^0 \int_0^{\sqrt{4-y^2}} xy\,dxdy = \frac{1}{2}\int_{-2}^0 [x^2y]_0^{\sqrt{4-y^2}}\ dy$

$= \frac{1}{2}\int_{-2}^0 (4-y^2)y\,dy = 1/2\ \int_{-2}^0 (4y-y^3)y\,dy = \frac{1}{2}\left[2y^2 - \frac{y^4}{4}\right]_{-2}^0 = \frac{1}{2}\big(0-(8-4)\big) = -2$