Logarithm and Exponential Functions

Logarithms are defined with respect to a particular base, but have a set of properties regardless of the base. The base may be any positive number, but there are three very commonly used bases; 10, 2 and e (footnote\(^1\)).

Definition

Let \( b \) be the base. If for a given number \( z \), \( b^x = z \), then \( x \) is said to be the logarithm (base \( b \)) of \( z \):

\[
x = \log_b z; \text{ } x \text{ is a logarithmic function of } z,
\]

\[
z = b^x; \text{ } z \text{ is an exponential function of } x.
\]

The exponential and logarithmic functions are mutually inverse.

Logarithms in base 10

If \( 10^x = a \), then \( x \) is said to be the logarithm (base 10) of \( a \).

<table>
<thead>
<tr>
<th>( \text{Base} )</th>
<th>( \text{Exponential} )</th>
<th>( \text{Logarithmic} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^{0.01} )</td>
<td>( 0.01 )</td>
<td>( \log_{10} 0.01 = -2 )</td>
</tr>
<tr>
<td>( 10^{0.1} )</td>
<td>( 0.1 )</td>
<td>( \log_{10} 0.1 = -1 )</td>
</tr>
<tr>
<td>( 10^0 )</td>
<td>( 1 )</td>
<td>( \log_{10} 1 = 0 )</td>
</tr>
<tr>
<td>( 10^1 )</td>
<td>( 10 )</td>
<td>( \log_{10} 10 = 1 )</td>
</tr>
<tr>
<td>( 10^{2} )</td>
<td>( 100 )</td>
<td>( \log_{10} 100 = 2 )</td>
</tr>
<tr>
<td>( 10^{3} )</td>
<td>( 1000 )</td>
<td>( \log_{10} 1000 = 3 )</td>
</tr>
</tbody>
</table>

Logarithms in base 2

If \( 2^x = a \), then \( x \) is said to be the logarithm (base 2) of \( a \).

<table>
<thead>
<tr>
<th>( \text{Base} )</th>
<th>( \text{Exponential} )</th>
<th>( \text{Logarithmic} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^{0.25} )</td>
<td>( 0.25 )</td>
<td>( \log_{2} 0.25 = -2 )</td>
</tr>
<tr>
<td>( 2^{0.5} )</td>
<td>( 0.5 )</td>
<td>( \log_{2} 0.5 = -1 )</td>
</tr>
<tr>
<td>( 2^0 )</td>
<td>( 1 )</td>
<td>( \log_{2} 1 = 0 )</td>
</tr>
<tr>
<td>( 2^1 )</td>
<td>( 2 )</td>
<td>( \log_{2} 2 = 1 )</td>
</tr>
<tr>
<td>( 2^2 )</td>
<td>( 4 )</td>
<td>( \log_{2} 4 = 2 )</td>
</tr>
<tr>
<td>( 2^3 )</td>
<td>( 8 )</td>
<td>( \log_{2} 8 = 3 )</td>
</tr>
</tbody>
</table>

\(^1e\) is the mathematical constant equal to 2.71828 18284 59045 23536 to 20 decimal places.
Logarithms in base $e$

If $e^x = a$, then $x$ is said to be the logarithm (base $e$) of $a$. $x$ can also be said to be the natural or Napierian logarithm, and is sometimes denoted by ln.

<table>
<thead>
<tr>
<th>$e^2$=0.1353352832</th>
<th>$\log_e 0.1353352832$ = -2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^1$=0.367879441171442</td>
<td>$\log_e 0.367879441171442$ = -1</td>
</tr>
<tr>
<td>$e^0$=1</td>
<td>$\log_e 1$ = 0</td>
</tr>
<tr>
<td>$e^4$=2.7182818284</td>
<td>$\log_e 2.7182818284$ = 1</td>
</tr>
<tr>
<td>$e^2$=7.38905609893065</td>
<td>$\log_e 7.38905609893065$ = 2</td>
</tr>
<tr>
<td>$e^3$=20.0855369231877</td>
<td>$\log_e 20.0855369231877$ = 3</td>
</tr>
</tbody>
</table>

Graphs of log(x)

If we plot log(x) for a range of values of $x$ and in the three most important bases then the following graphs are obtained.

Although the graphs are different for different bases, they have a number of characteristics in common:

(i) they all pass through the point $(1,0)$; log (1)=0 in all bases
(ii) the graph reaches the limit of $-\infty$ as $x$ tends to zero
(iii) the graphs “flatten out” as $x$ tends to $\infty$.  

![Graphs of log x](image)
Changing base of Logarithms

Log graphs essentially have the same shape; multiplying the log graph in one base by a number gives the log graph in another base:

\[ \log_c x = \frac{\log_b x}{\log_b c}. \]

For example with \( x = 4, b = 2 \) and \( c = 10 \):

\[ \log_2 4 = \frac{\log_{10} 4}{\log_{10} 2} = \frac{0.6020599913}{0.3010299557} = 2 \]

Properties are true for logarithms in any base.

These properties made logarithms useful in the days before widespread use of computers.

(i)
\[ \log(xy) = \log(x) + \log(y) \]

For example
\[ \log_2(8) = \log_2(4 \times 2) = \log_2(4) + \log_2(2) = 2 + 1 = 3. \]

In the days before there was a widespread availability of computers, for a difficult multiplication (say \( x \) and \( y \)) first the logs of the two numbers would be looked up (giving \( \log(x) \) and \( \log(y) \)). He numbers would be added ( to give \( \log(x) + \log(y) \) which is equal to \( \log(xy) \). By taking the antilogarithm (of \( \log(xy) \)) from the same book of tables, the value of \( xy \) is obtained.

(ii)
\[ \log(x/y) = \log(x) - \log(y) \]

For example
\[ \log_2(2) = \log_2(4/2) = \log_2(4) - \log_2(2) = 2 - 1 = 1. \]

(iii)
\[ \log(x/y) = \log(x) - \log(y) \]

For example
\[ \log_2(2) = \log_2(4/2) = \log_2(4) - \log_2(2) = 2 - 1 = 1. \]

(iv)
\[ \log(1/y) = -\log(y) \]

For example
\[ \log_{10}\left(\frac{1}{100}\right) = -2 = -\log_{10}(100). \]
(v) \[ \log(x^p) = p \log(x) \]

For example
\[ \log_{10}(1000) = \log_{10}(10^3) = 3 \log_{10}(10) = 3 \times 1 = 3. \]

**Graphs of** $b^x$

If we plot $b^x$ for a range of values of $x$ and in the three most important bases then the following graphs are obtained.

The green graph is $10^x$, the red graph is $e^x$ and the blue graph is $2^x$. Since $10^x$ grows much faster than the other graphs, then the two other graphs are shown below.