Logarithm and Exponential Functions

Logarithms are defined with respect to a particular *base*, but have a set of properties regardless of the base. The base may be any positive number, but there are three very commonly used bases; 10, 2 and *e* (footnote[1]).

Definition

Let *b* be the base. If for a given number *z*, $b^x = z$, then *x* is said to be the logarithm (base *b*) of *z*:

$x = \log_b z$; x is a logarithmic function of z,

$z = b^x$; z is an exponential function of x .

The exponential and logarithmic functions are mutually inverse.

Logarithms in base 10

If $10^x = a$, then *x* is said to be the logarithm (base 10) of *a*.

$10^{-2}=0.01$	$\log_{10} 0.01 = -2$
$10^{-1}=0.1$	$\log_{10} 0.1 = -1$
$10^0=1$	$\log_{10} 1 = 0$
$10^1=10$	$\log_{10} 10 = 1$
$10^2=100$	$\log_{10} 100 = 2$
$10^3=1000$	$\log_{10} 1000 = 3$

Logarithms in base 2

If $2^x = a$, then *x* is said to be the logarithm (base 2) of *a*.

$2^{-2}=0.25$	$\log_2 0.25 = -2$
$2^{-1}=0.5$	$\log_2 0.5 = -1$
$2^0=1$	$\log_2 1 = 0$
$2^1=2$	$\log_2 2 = 1$
$2^2=4$	$\log_2 4 = 2$
$2^3=8$	$\log_2 8 = 3$

[1] *e* is the mathematical constant equal to 2.71828 18284 59045 23536 to 20 decimal places.

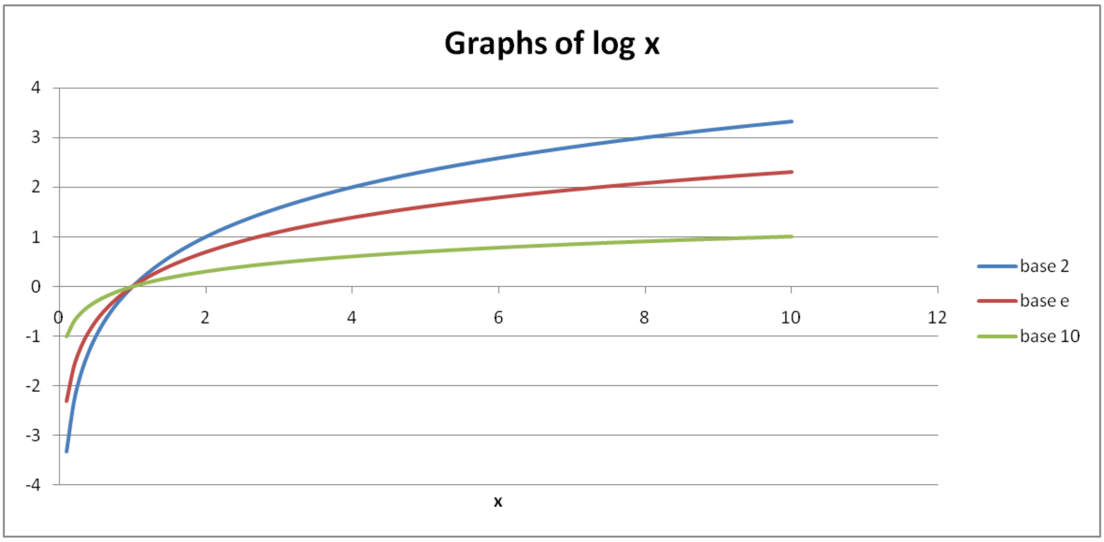
Logarithms in base e

If $e^x = a$, then x is said to be the logarithm (base e) of a, x can also be said to be be the natural or Napierian logarithm, and is sometimes denoted by ln.

e^{-2}=0.1353352832	$\log_e$ 0.1353352832= -2
e^{-1}=0.367879441171442	$\log_e$ 0.367879441171442= -1
e^0=1	$\log_e$ 1 = 0
e^1=2.71828 18284	$\log_e$ 2.71828 18284= 1
e^2=7.38905609893065	$\log_e$ 7.38905609893065= 2
e^3=20.0855369231877	$\log_e$ 20.0855369231877 = 3

Graphs of log(x)

If we plot log(x) for a range of values of x and in the three most important bases then the following graphs are obtained.



Although the graphs are different for different bases, they have a number of characteristics in common:

(i) they all pass through the point (1,0); log (1)=0 in all bases
(ii) the graph reaches the limit of $-\infty$ as x tends to zero
(iii) the graphs "flatten out" as x tends to ∞ .

Changing base of Logarithms

Log graphs essentially have the same shape; multiplying the log graph in one base by a number gives the log graph in another base:

$$\log_b x = \frac{\log_c x}{\log_c b}.$$

For example with $x = 4,\ b = 2$ and $c = 10$: $\log_2 4 = \frac{\log_{10} 4}{\log_{10} 2} = \frac{0.6020599913}{0.3010299957} = 2$

Properties are true for logarithms in any base.

These properties made logarithms useful in the days before widespread use of computers.

(i)
$\log(xy) = \log(x) + \log(y)$

For example
$\log_2(8) = \log_2(4 \times 2) = \log_2(4) + \log_2(2) = 2 + 1 = 3.$

In the days before there was a widespread availability of computers, for a difficult multiplication (say *x* and *y*) first the logs of the two numbers would be looked up (giving $\log(x)$ and $\log(y)$). He numbers would be added (to give $\log(x) + \log(y)$, which is equal to $\log(xy)$. By taking the *antilogarithm* (of $\log(xy)$) from the same book of tables, the value of xy is obtained.

(ii)
$\log(x/y) = \log(x) - \log(y)$

For example
$\log_2(2) = \log_2(4/2) = \log_2(4) - \log_2(2) = 2 - 1 = 1.$

(iii)
$\log(x/y) = \log(x) - \log(y)$

For example
$\log_2(2) = \log_2(4/2) = \log_2(4) - \log_2(2) = 2 - 1 = 1.$

(iv)
$\log(1/y) = -\log(y)$

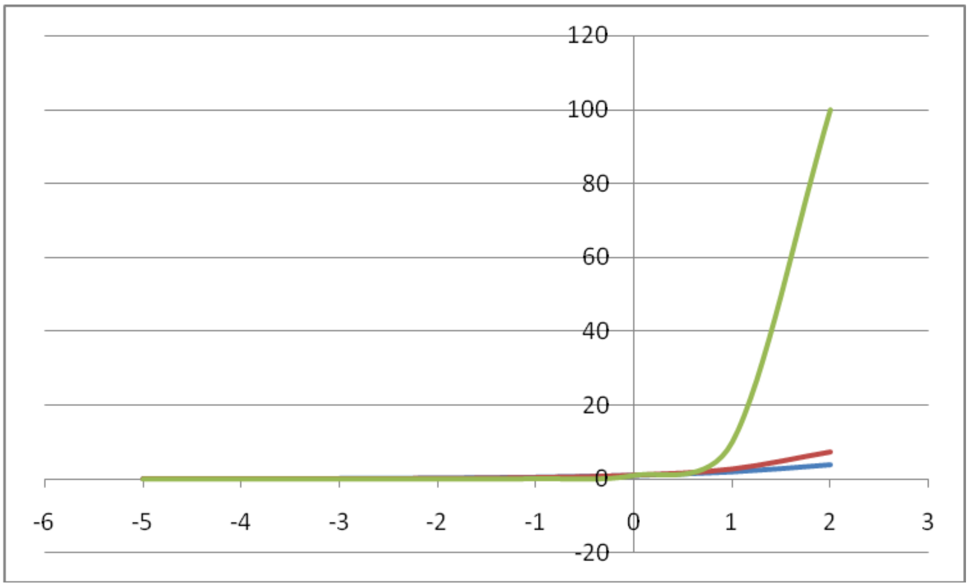
For example
$\log_{10}\left(\frac{1}{100}\right) = -2 = -\log_{10}(100).$

(v)
$\log(x^p) = p \log(x)$

For example
$\log_{10}(1000) = \log_{10}(10^3) = 3\log_{10}(10) = 3 \times 1 = 3$.

Graphs of b^x

If we plot b^x for a range of values of *x* and in the three most important bases then the following graphs are obtained.



The green graph is 10^x, the red graph is e^x and the blue graph is 2^x. Since 10^x grows much faster than the other graphs, then the two other graphs are shown below.