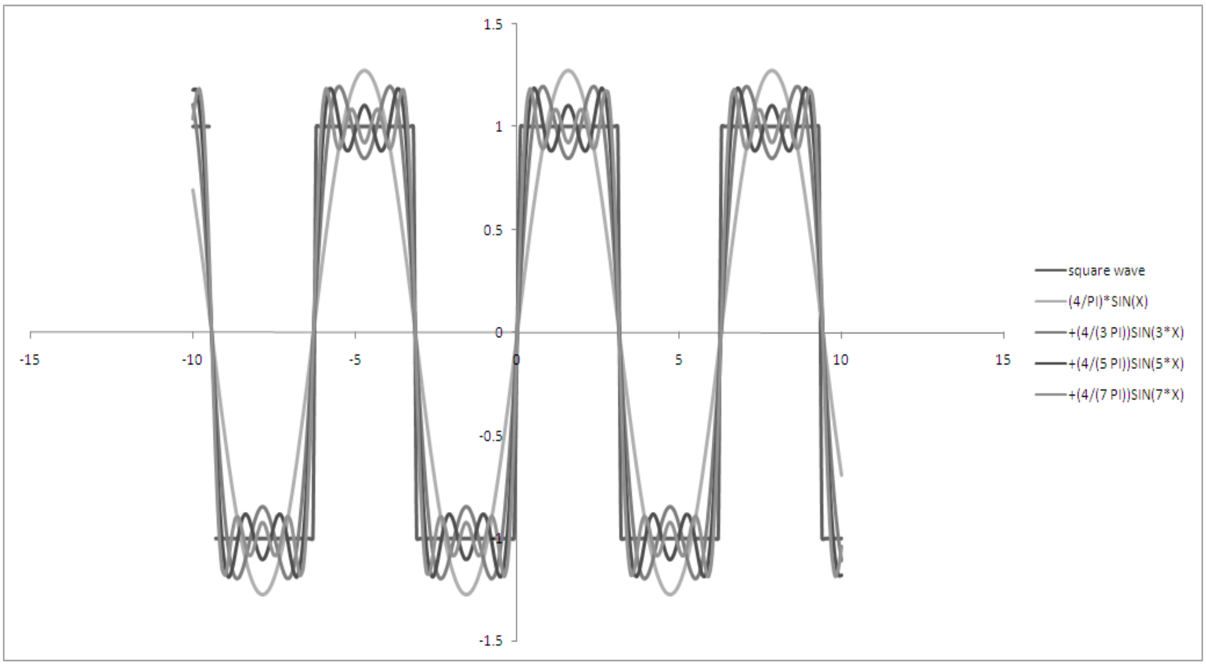
Fourier Series

A Fourier series represents a periodic function[1] as a sum of sinusoidal functions. The sinusoidal functions begin at the large wavelengths (low frequencies) and continue progressively to the short wavelengths (high frequencies). The illustration below shows how the first few terms of the corresponding Fourier series approximate a square wave.



Since the sine and cosine functions have period 2π, then it's natural to start by defining Fourier series defined on the initial period $(-\pi\,,\pi)$ or $(0\,,2\pi)$. Fourier series can be determined for any period T simply by multiplying the axial coordinate by $\frac{T}{2\pi}$.

Determining the Fourier Series for functions that are periodic on $(-\pi\,,\pi)$

Let $f(t)$ be a (periodic) function defined on the domain $-\pi < t < \pi$. The Fourier series for $f(t)$ is a weighted sum of sine and cosine functions as follows:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin\, nt$$

where the a_n and b_n are constants, which can be determined by integration[2]:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt\,,$$

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[1] Periodic Functions

[2] Integration

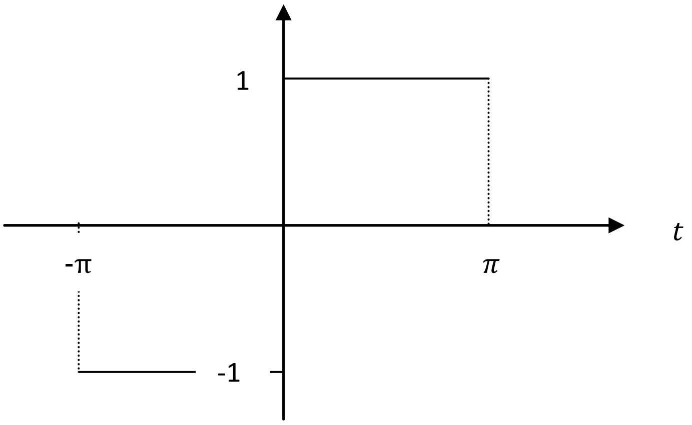
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n\pi t)\,dt\,,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\pi t)\,dt\,.$$

Hence in order to determine the coefficients of a Fourier series, we need to complete the above integrals. However, there are a number of shortcuts, which can save time. Firstly the first term of the Fourier series, $\frac{a_0}{2}$, is the mean or average value of $f(t)$ in $(-\pi\,,\pi)$. Secondly – noting that the sine function is *odd* and the cosine function is *even*[3]– if $f(t)$ is odd then the Fourier series consists only of sin terms ($a_n = 0$) and if $f(t)$ is even then the Fourier series consists of only cosine terms.

Example 1

Find the Fourier series for the following periodic function with period 2π and defined in the domain



$$f(t) = \begin{cases} -1 & -\pi < t \le 0, \\ 1 & 0 \le t < \pi. \end{cases}$$

Solution

Firstly we note that the function is odd; $a_n = 0$, for all n.

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[3] Even and Odd Functions

We set out to find the values of b_n using the standard formula

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$, which involves an integration over the domain *[-π,π].*

In cases when there are discontinuities in the function *f(t)*, it is convenient to divide the domain of integration at the discontinuities. In this case the discontinuity is at *t=0,* hence we split the integral as follows:

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}[\int_{-\pi}^{0} f(t)\sin(nt)\,dt + \int_{0}^{\pi} f(t)\sin(nt)\,dt.$

We can now insert the actual function *f(t)* in each sub-domain

$$b_n = \frac{1}{\pi}[\int_{-\pi}^{0}(-1)\sin(nt)\,dt + \int_{0}^{\pi}(1)\sin(nt)\,dt\,]$$

$$= \frac{1}{\pi}[-\int_{-\pi}^{0}\sin(nt)\,dt + \int_{0}^{\pi}\sin(nt)\,dt\,]$$

$$= \frac{1}{\pi}\{-\left[-\frac{1}{n}\cos(nt)\right]_{-\pi}^{0} + \left[-\frac{1}{n}\cos(nt)\right]_{0}^{\pi}\}$$

$$= \frac{1}{n\pi}\{[\cos(n\pi t)]_{-\pi}^{0} - [\cos(n\pi t)]_{0}^{\pi}\} = \begin{cases} 0 \text{ if } n \text{ is even} \\ \frac{4}{n\pi} \text{ if } n \text{ is odd} \end{cases}$$

It is useful to form a table of values of a_n and b_n for the first few values of *n,* as follows.

n	0	1	2	3	4	5
a_n	0	0	0	0	0	0
b_n	-	$\frac{1}{\pi}$	0	$\frac{1}{3\pi}$	0	$\frac{1}{5\pi}$

Hence we may write

$$f(t) = \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\sin 5t + \cdots.$$

The graph on the first page shows the sequence of partial sums of the example above: $\frac{4}{\pi}\sin t$, $\frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t$, $\frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\sin 5t, \dots.$ It is also shown more interactively on a spreadsheet[4].

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[4] Google spreadsheet showing sequence of partial sums for example 1.

There are a number of ways the initial definition and determination of Fourier series can be made more complicated and we will consider two of these. Firstly the domain of the periodic signal does not have to be $(-\pi, \pi)$, let us assume that it has a general length $2L$ and lies in the domain $(-L, L)$.

Let $f(t)$ be a (periodic) function defined on the domain $-L < t < L$. The Fourier series for $f(t)$ then has the following form:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt\,,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{n\pi t}{L})\,dt\,,$$

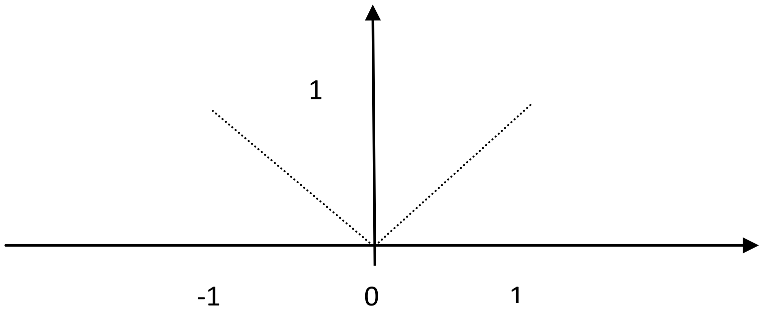
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(\frac{n\pi t}{L})\,dt\,.$$

Secondly, when non-constant functions are used then we usually need to use product integration in order to evaluate it.

Example 1

Find the Fourier series for the following periodic *saw tooth* function with period 2 and defined in the domain [-1,1] as follows:

$$f(t) = \{\begin{matrix} -x & -1 < t \le 0, \\ x & 0 \le t < 1. \end{matrix}$$



Solution

Firstly we note that the function is even; $b_n = 0$, for all *n*.

To determine a_0 we note that the average value of the function $f(t)$ is $\frac{1}{2}$; $a_0 = \frac{1}{2}$.

We set out to find the values of a_n using the standard formula

$$a_n = \int_{-1}^{1} f(t)\cos(n\pi t)\,dt\,,$$

which involves an integration over the domain *[-1, 1].*

In this case there is a discontinuity at *t=0,* hence we split the integral as follows:

$$a_n = \int_{-1}^{1} f(t)\cos(n\pi t)\,dt = \int_{-1}^{0} f(t)\cos(n\pi t)\,dt + \int_{0}^{1} f(t)\cos(n\pi t)\,dt$$

$$= \int_{-1}^{0} (-t)\cos(n\pi t)\,dt + \int_{0}^{1} t\cos(n\pi t)\,dt$$

$$= -\int_{-1}^{0} t\ \cos(n\pi t)\,dt + \int_{0}^{1} t\cos(n\pi t)\,dt\,.$$

Using integration by parts, we note that

$$\int t\cos(at) = \ [\ \frac{1}{a}\sin(at)\ \ t\,] - \int \frac{1}{a}\sin(at)1\,dt = [\frac{1}{a}\sin(at)\ \ t\,] + \ \frac{1}{a^2}\cos(at)\,]\,.$$

Hence

$$-\int_{-1}^{0} t\ \cos(n\pi t)\,dt + \int_{0}^{1} t\cos(n\pi t)\,dt$$

$$= -[\frac{1}{n\pi}\sin(n\pi t) + \frac{1}{(n\pi)^2}\ \cos\,(n\pi t)]_{-1}^{0} + [\frac{1}{n\pi}\sin(n\pi t) + \frac{1}{(n\pi)^2}\ \cos\,(n\pi t)]_{0}^{1}$$

$$= -\,(\frac{1}{(n\pi)^2} + \frac{1}{(n\pi)^2}) + (-\,\frac{1}{(n\pi)^2} - \frac{1}{(n\pi)^2}) = \frac{-4}{(n\pi)^2}\ \text{for } n \text{ odd}$$

$$= -\,(\frac{1}{(n\pi)^2} - \frac{1}{(n\pi)^2}) + (-\,\frac{1}{(n\pi)^2} + \frac{1}{(n\pi)^2}) = 0\ \text{for } n \text{ even.}$$

It is useful to form a table of values of a_n and b_n for the first few values of n, as follows.

n	0	1	2	3	4	5	6
a_n	½	$\frac{-4}{\pi^2}$	0	$\frac{-4}{9\pi^2}$	0	$\frac{-4}{9\pi^2}$	0
b_n	0	0	0	0	0	0	0

Hence

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\cos(\pi t) - \frac{4}{9\pi^2}\cos(3\pi t) - \frac{4}{25\pi^2}\cos(5\pi t) - \cdots$$

The approximation of the saw tooth function by a fourier series is demonstrated on a linked Google spreadsheet[5].

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[5] Saw tooth function approximated by a Fourier series