Complex Numbers

Complex numbers represent an extension on the concept of real numbers. Whilst our understanding of real numbers is more practical, without complex numbers vast areas of science and engineering would be undiscovered, or not understood.

For complex numbers to exist, we must only allow one simple concept; that the square-root of -1 has a value, this is denotes i in mathematics, but it is often denoted j in science and engineering. In this document we will use the following notation and definition of the unit imaginary number:

$$\sqrt{-1} = i.$$

A real number is any real-valued multiple of the unit real number 1. Similarly, an imaginary number is any multiple of the unit imaginary number i.

For example the real number $6.0 = 6.0 \times 1$ and the imaginary number $4.0i = 4.0 \times i$.

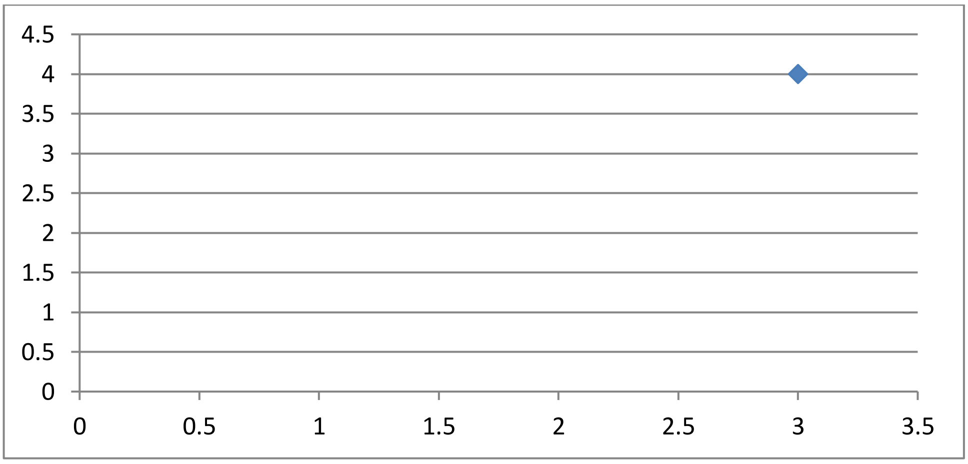
A complex number generally consists of a real and an imaginary part, and can be written $z = x + iy$, where z is a complex number and x and y are real numbers; x is said to be the real part of z (Re(z)=x) and y is said to be the imaginary part of z (Im(z)=y).

For example $3 + 4i$ is a complex number. $\mathrm{Re}(3 + 4i) = 3$ and $\mathrm{Im}(3 + 4i) = 4$.

Argand Diagram

A complex number z can be represented graphically on an Argand diagram.

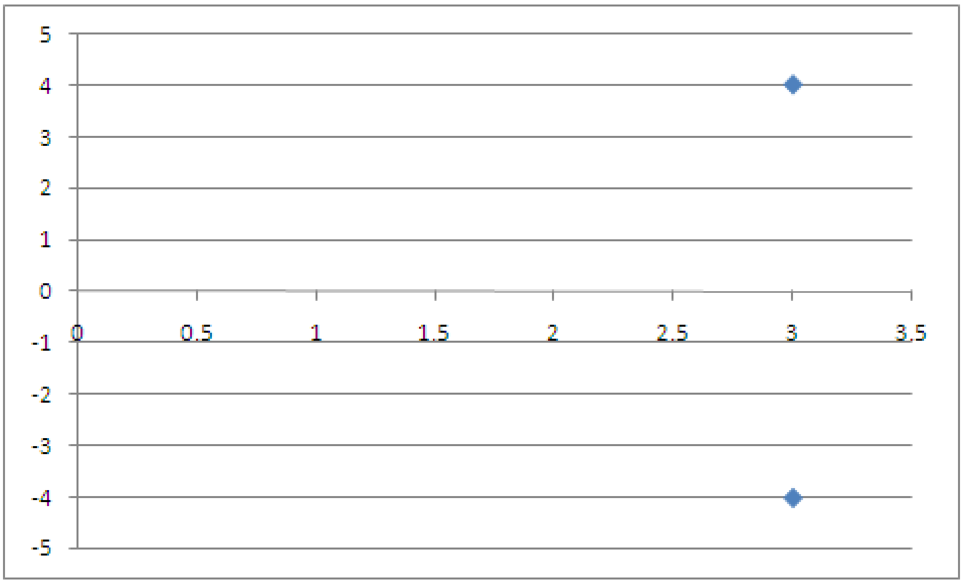
The following graph shows the number $3 + 4i$ represented on an Argand diagram.



Complex Conjugate

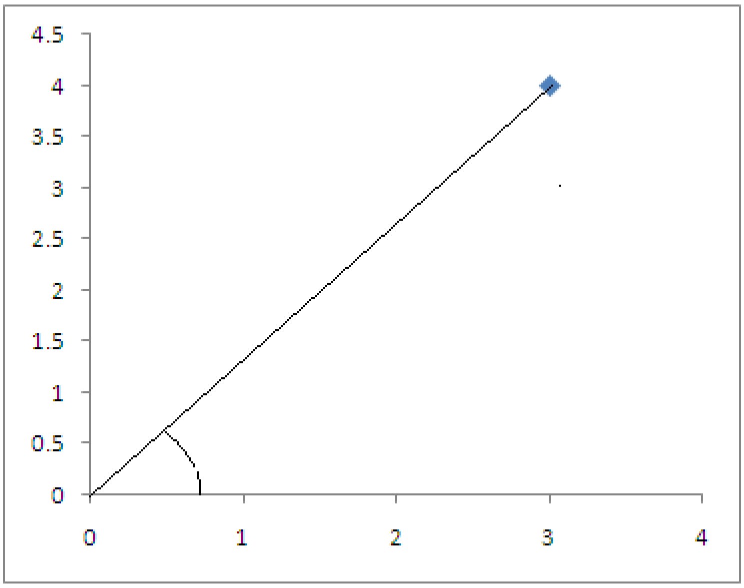
The complex conjugate of the complex number $z = x + iy$ is denoted $\bar{z}$ and it is defined so that $\bar{z} = x - iy$. On an Argand diagram, a complex number and its complex conjugate are mutually reflective in the horizontal axis.

For example $3+4i$ has the complex conjugate $3-4i$. The two complex numbers are illustrated on the following Argand diagram.



Modulus and Argument

A complex number has a modulus (or size) and an argument (or angle).



The diagram above illustrates the modulus and argument of $3+4i$; the modulus is length of the line between the origin to the point representing the complex number and the argument is the angle that that the line makes with the horizontal.

In general for a complex number $z = x + iy$, it can be shown from elementary trigonometry that the modulus is defined as $|z| = \sqrt{x^2 + y^2}$ and the argument is defined by $\tan \varphi = \frac{y}{x}$.

For example

$$|3+4i| = \sqrt{3^2+4^2} = \sqrt{25} = 5$$

$\tan\varphi = \frac{4}{3}$, hence $\arg(3+4i) = \varphi = 0.9273$ radians or $53.13°$

[Further information on measures of angle (radians and degrees)]

A complex number z, with modulus $|z|$ and argument φ can be written

$$z = |z|\cos\varphi + i|z|\sin\varphi$$

For example $3+4i = 5\cos(0.9273) + i5\sin(0.9273)$.

Multiplying a complex number by its complex conjugate gives a real number equal to the square of the modulus. In general

$$z\bar{z} = |z|^2 \, .$$

For example $(3+4i)(3-4i) = 9 - 12i + 12i - 16i^2 = 9 + 16 = 25 = 5^2$.

Exponential Notation

Let us define the complex exponential as follows

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

A complex number z, with modulus $|z|$ and argument φ can be written

$$z = |z|e^{i\varphi}$$

Spreadsheet

The accompanying spreadsheet plots the Argand diagram of a complex number and computes the modulus and argument,

Complex Numbers

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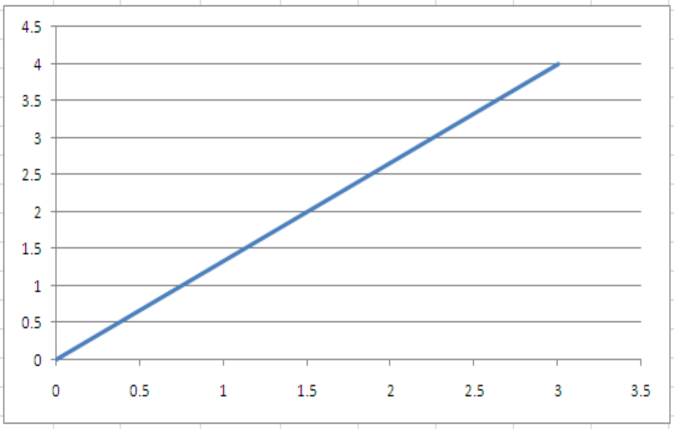
Complex Number Notes

complex number

3 + 4 i

place the complex number in the yellow area

Argand Diagram



modulus= 5

angle= 0.927295 radians

53.1301 degrees