Big O Notation in Mathematics

In mathematics (big) O or ‘order’ notation describes the behaviour of a function at (a point) zero or as it approaches infinity. With O notation the function \( f(x) \) is usually simplified, for example to a power of \( x \) or an exponential, logarithm\(^1\), factorial\(^2\) function, or a combination of these functions. Big O notation does not approximate the original function \( f(x) \), but rather it models its essential behaviour. The same notation is extended to computing in which it is used to give a shorthand measure of the efficiency of algorithms or the memory requirements of computer programs\(^3\). In the cross-over subject of numerical methods\(^4\), both the mathematical and the computational applications of O notation are used. In this document we will develop an understanding of O notation in mathematics.

O notation for representing a function at zero

In this section we first consider O notation for functions that are finite at \( x = 0 \) and then progress to consider functions that are infinite at \( x = 0 \). Let us start by considering typical functions that are used on O notation and are finite at \( x = 0 \). The following graph shows the functions \( 1, x, x^2 \) and \( x^3 \).

![Graph showing functions 1, x, x^2, x^3](image)

The graph verified that for small values of \( x \), \( 1 \gg x \gg x^2 \gg x^3 \).

**Example 1**

Let us begin by looking at a simple example of using O notation. Let \( f(x) = 1 + x + x^2 + x^3 \). Since \( 1 \gg x \gg x^2 \gg x^3 \), then \( f(x) \approx 1 \) and \( f(x) = O(1) \) as \( x \to 0 \).

**Example 2**

In the second example let \( f(x) = 2 + x + x^2 + x^3 \). In this case \( f(x) \approx 2 \) and \( f(x) = O(2) \) as \( x \to 0 \). However, it is the convention in O notation to ignore the coefficient of the standard function and in this case \( (x) = O(1) \).

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2 Factorial
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4 www.numerical-methods.com
Example 3

In the third example let \( f(x) = -3x^2 + x^3 \). In this case \( f(x) \approx -3x^2 \) and \( f(x) = O(x^2) \) as \( x \to 0 \). In the answer the coefficient ‘-3’ is dropped, following the convention of \( O \) notation.

Example 4

In the third example let \( f(x) = 2x + 1000000x^2 \). The answer is \( f(x) = O(x) \) as \( x \to 0 \). Note that although the coefficient of \( x^2 \) is large, when \( x \) is very small the ‘\( 2x \)’ term still dominates and the method for determining the \( O \) notation representation of the function is unchanged.

Example 5

In this example we consider the function \( f(x) = \sin(x) \) at \( x = 0 \). In order to write \( f(x) \) in terms of the standard functions then we first consider the Maclaurin\(^5\) expansion for \( \sin(x) \):

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.
\]

A truncation of the Maclaurin expansion provides an approximation to the function for small values of \( x \). In \( O \) notation only one term is used for functions of one variable and since for small values of \( x \) the \( x \) is the dominant term then we may write

\[
\sin(x) \approx x
\]

for small values of \( x \). The following graph illustrates the point.

Using \( O \) notation we may write

\[
\sin(x) = O(x) \text{ as } x \to 0.
\]

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\(^5\) Maclaurin Series
Example 6

In this example let us consider the function \( f(x) = 1 - \cos(x) \) as \( x \to 0 \).

Let us first state the Maclaurin\(^6\) expansion for \( f(x) \):

\[
1 - \cos(x) = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots
\]

Since we only choose the dominant term on \( O \) notation then we can consider the approximation

\[
1 - \cos(x) \approx \frac{x^2}{2},
\]

as illustrated in the following graph.

Using \( O \) notation we may write:

\[
1 - \cos(x) = O(x^2).
\]

Example 7

For the final example in this section, let us consider a function with a number of components:

\[
f(x) = 3 + 2\sin(x) - x^2 + \cos(x) + e^{2x} + 5x^4.
\]

The answer to this is simply \( f(x) = O(1) \). This example shows how the \( O \) notation gives a concise representation of the function.

**O notation for representing a function that is infinite at zero**

Typical functions that are infinite at \( x = 0 \) and are used in \( O \) notation are the negative powers of \( x \) and logarithms of \( x \). The following graph compares \( \log(x) \), \( \frac{1}{x} \) and \( \frac{1}{x^2} \) as \( x \to 0 \).

\(^6\) Maclaurin Series
From the graph it can be seen that for small values of $x$, \( \frac{1}{x^2} \gg \frac{1}{x} \gg \log(x) \).

[Note that the base of the logarithm function is unspecified. The reason for this is that the logarithm function in one base is equal to a constant multiplied by a logarithm in another base\(^7\), and coefficients can be ignored in $O$ notation and hence it is acceptable (and desirable) not to specify the base of the logarithm when applying $O$ notation.]

**Example 7**

In the first example of functions that are infinite when $x = 0$. Let $f(x) = \log(x) + \frac{1}{x^3} + \frac{1}{x}$. As $x \to 0$ the $\frac{1}{x^3}$ term is dominant and hence in this case $f(x) = O\left(\frac{1}{x^3}\right)$.

**Example 8**

Let $f(x) = 100 + e^x + 2 \ln(x)$. In this case the term that is infinite at $x = 0$ and hence $f(x) = O(\ln x)$. Since we can ignore the base of the logarithmic function then the better answer is simply $f(x) = O(\log x)$.

**$O$ notation for representing a function at infinity**

In this section we consider the $O$ representation for a function $f(x)$ as $x \to \infty$. As mentioned earlier, $O$ notation is used in computing. The use of $O$ notation in computing is an application of this in which the focus is on the memory requirements and processing time as the amount of data handled increases\(^8\).

The following graph demonstrates the relative growth rates of typical functions that are used in $O$ notation. The functions $f(x) = 1, \log x, x, \log x, x^2, x^3, 2^x$ are considered. In order to compare the functions are multiplied by a constant so that they are equal at $x = 4$.

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\(^7\) Logarithm and Exponential Functions
\(^8\) Big O Notation in Computing
The graphs show that for large values of \( x \), \( 2^x \gg x^3 \gg x^2 \gg x \log x \gg x \gg \log x \gg 1 \). In words we can say that generally the higher the power of \( x \) the stronger the growth; \( \ldots x^5 \gg x^4 \gg x^3 \gg x^2 \gg x \gg 1 \). The exponential function has a stronger growth rate than any powers of \( x \) and the larger the base of the exponential, the stronger the growth (for example \( 3^x \gg e^x \gg 2^x \) for large values of \( x \)). \( \log x \) has weaker growth than \( x \).

**Example 1**

In the first example let \( f(x) = 3x^3 + 2x^2 - 5x + 4 \). In order to represent \( f(x) \) using \( O \) notation we first observe that the \( 3x^3 \) is the term with the strongest growth as \( x \to \infty \). In \( O \) notation we ignore the coefficient, so the answer is \( f(x) = O(x^3) \).

**Example 2**

In this example let \( f(x) = x - 2x \ln x + 5 \). In order to represent \( f(x) \) using \( O \) notation we first observe that the \( -2x \ln x \) is the term with the strongest growth as \( x \to \infty \). In \( O \) notation we ignore the coefficient, so the answer is \( f(x) = O(x \ln x) \).

**Example 3**

Let \( f(x) = x^5 - \log x + 2^x \). In order to represent \( f(x) \) using \( O \) notation we first observe that the \( 2^x \) is the term with the strongest growth as \( x \to \infty \). In \( O \) notation we ignore the coefficient, so the answer is \( f(x) = O(2^x) \).

**Example 4**

Let \( f(x) = \ln(2x^3 + \sin x - x + 1) \). In order to represent \( f(x) \) using \( O \) notation we first observe that the outer \( \ln \) grows as its argument grows. The term within the argument with the strongest growth is the \( 2x^3 \) term. Hence we may state that for large values of \( x \): \( f(x) \sim \ln(2x^3) \). Using the properties of logarithms \( \ln(2x^3) = \ln 2 + 3 \ln x \), so \( f(x) \sim \ln 2 + 3 \ln x \). The \( \ln \) function is the strongest growing term and in \( O \) notation we ignore the coefficient and the base and so the answer is \( f(x) = O(\ln x) \).