Solving a pair of simultaneous linear equations

A pair of linear simultaneous equations have the form:

$$a_{11}x + a_{12}y = b_1 \qquad (1)$$
$$a_{21}x + a_{22}y = b_2 \qquad (2)$$

where a_{11}, a_{12}, a_{21}, and a_{22} are constant coefficients and b_1 and b_2 are constants. The task is to find the values of x and y.

Example 1

For example let two apples and one orange cost 7 and let three apples and two oranges cost 12, find the individual cost of an apple and an orange.

We can formulate this problem in the form of two simultaneous equations as follows:

$$2x + y = 7$$
$$3x + 2y = 12$$

where x and y represent the unknown individual cost of the apples and oranges respectively.

There are many, or a family of solutions to each individual equation. For the example $x=1, y=5$ or $x=3, y=1$ are possible solutions to the first equation, but they are not solutions to the second. The solution to the problem are the values of x and y that solve both equations *simultaneously*.

If we substitute $x=2$, $y=3$, it can be seen that these values satisfy both equations and this is the solution.

There are two special cases of simultaneous equations.

1. No unique solution

If the second equation merely repeats the first equation then the second equation gives us no further information. So we continue to have a family of

Example 2

$$2x + y = 7$$
$$4x + 2y = 14$$

The second equation merely repeats the first, doubling all the constants. All solutions have the form $x=a$, $y=7-2a$, where a is any number.

solutions:

2. No solution at all

If the second equation effectively contradicts the first equation then there is no solution.

Example 3

$$2x + y = 7$$
$$4x + 2y = 13$$

There is no solution; no values of x and y can solve both equations simultaneously.

Matrix-vector form

A set of simultaneous linear equations can also be written in matrix vector form. We may write the original equations (1),(2) in the form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Example 4

The simultaneous equation of example 1

$$2x + y = 7$$
$$3x + 2y = 12$$

can be written in the form:

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 12 \end{pmatrix}$$

Solution method

How do we find the unique solution in the majority of cases there is one? There are actually four distinct methods:

1. By plotting a graph,
2. By elimination,
3. By substitution,
4. By the use of matrices and vectors.