Vector Norm and Normalisation

The *norm* of a vector is a measure of its size. There are several different types of norms. The norm of a vector $\underline{x}$ is denoted $\|\underline{x}\|$. The type of norm is indicated by a subscript.

2-norm

The most common norm is the Euclidean or 2-norm, denoted$\| \quad \|_2$.

For example for a 2-vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\left\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\|_2 = \sqrt[2]{{x_1}^2 + {x_2}^2}$, which can be interpreted as the distance from the origin to a point with coordinates x_1 and x_2.

For a 3-vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\left\|\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right\|_2 = \sqrt[2]{{x_1}^2 + {x_2}^2 + {x_3}^2}$, which can be interpreted as the distance from the origin to a point with coordinates (x_1, x_2, x_3).

Generalising to an n-vector $\underline{x}$,

$$\|\underline{x}\|_2 = \sqrt[2]{\sum_{i=1}^{n} {x_i}^2} \, .$$

1-norm

For an n-vector $\underline{x}$, the 1-norm is defined as follows:

$$\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

p-norm

Generalising the 1- and 2- norms to a general value p for an n-vector $\underline{x}$ gives the p-norm:

$$\|\underline{x}\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p} \, .$$

∞-norm

For an n-vector $\underline{x}$, the ∞-norm is defined as follows:

$$\|\underline{x}\|_\infty = \max(\, |x_1|, |x_2|, \dots , \qquad |x_n|).$$

Examples

Find the 1-norm, 2-norm, 3-norm and ∞-norm of the vector $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.

The 1-norm is equal to $|1|+|-2| + |3| = 6$.

The 2-norm is equal to $\sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14} = 3.741657$

The 3-norm is equal to $\sqrt[3]{|1|^3 + |-2|^3 + |3|^3} = \sqrt[3]{36} = 3.301927$

The ∞-norm is equal to the largest element of $\underline{x}$; 3.

Normalisation

Normalisation is the process of scaling a vector so that its norm is unity. This can be carried out in any norm and can be achieved by simply dividing the vector by its norm.

Examples

Normalise the vector $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ with respect to the 1-norm, 2-norm, 3-norm and ∞-norm,

In the 1-norm the normalised vector is $\frac{1}{6}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.166667 \\ -0.333333 \\ 0.5 \end{pmatrix}$.

In the 2-norm the normalised vector is $\frac{1}{3.741657}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.267261 \\ -0.53452 \\ 0.801784 \end{pmatrix}$.

In the 3-norm the normalised vector is $\frac{1}{3.301927}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.302853 \\ -0.60571 \\ 0.90856 \end{pmatrix}$.

In the 3-norm the normalised vector is $\frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0.333333 \\ -0.666667 \\ 1 \end{pmatrix}$.