Standard functions applied to complex numbers

The standard mathematical functions – such as those found on a typical scientific calculator[1] – are thought of as being applicable to real numbers in the first instance. However, their definition also extends to complex arguments. In this document we explore the application of standard functions to complex numbers[2]. This techniques in this document also require knowledge of complex arithmetic[3].

For the purposes of this work, we will use the notation that z is a complex number that can be written in the form $z = a + ib = Ae^{i\theta}$.

logarithm base e	$\log_e(z) = \log_e\left(Ae^{i\theta}\right) = \log_e(A) + \log_e\left(e^{i\theta}\right) = \log_e(A) + i\theta$
exponential	$e^z = e^{(a+ib)} = e^a e^{ib} = e^a(\cos(b) + i\sin(b))$
Complex power	$z^w = \exp(\log z^w) = \exp\,(w\log z)$ (note exp and log for complex arguments are already defined)
Hyperbolic functions	Are defined in terms of the exponential above.
Trigonometric functions	We can derive the sine and cosine of complex numbers by using the expressions for the sines and cosines of sums[4]: $$\sin(\mathrm{z}) = \sin(\mathrm{a} + \mathrm{ib}) = \sin(\mathrm{a})\cos(\mathrm{ib}) + \cos(\mathrm{a})\sin(\mathrm{ib})\,.$$ $$\cos(\mathrm{z}) = \cos(\mathrm{a})\cos(\mathrm{ib}) - \sin(\mathrm{a})\sin(\mathrm{ib})\,.$$ Also $\sin(\varphi) = \frac{1}{2i}\left(e^{i\varphi} - e^{i\varphi}\right), \cos(\varphi) = \frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right)$ Hence $\sin\,(ib) = \frac{1}{2i}\left(e^{i(ib)} - e^{-i(ib)}\right) = \frac{1}{2i}\left(e^{-b} - e^{b}\right) = i\sinh\,(b)$ Hence $\cos(ib) = \frac{1}{2}\left(e^{i(ib)} + e^{-i(ib)}\right) = \frac{1}{2}\left(e^{-b} + e^{b}\right) = \cosh\,(b)$ Hence $$\sin \mathrm{z} = \sin(\mathrm{a})\cos\mathrm{h}(\mathrm{b}) + i\cos(\mathrm{a})\sinh\,(\mathrm{b})$$ and $$\cos \mathrm{z} = \cos(\mathrm{a})\cosh(\mathrm{b}) - i\sin(\mathrm{a})\sinh(\mathrm{b})\,.$$ Note that $\tan \mathrm{z} = \frac{\sin \mathrm{z}}{\cos \mathrm{z}}$.

[1] Standard Mathematical Functions: Windows Scientific Calculator
[2] Complex Numbers
[3] Complex Arithmetic
[4] Sine, Cosine and Tangent of Sum

Examples

In the following examples we will be using the complex number 3+4j, which has a modulus of 5 and an argument (angle) of 0.9273 radians

1. Logarithm

$$\log_e(3+4i) = log_e 5 + i0.9273 = 1.6094 + i0.9273$$

2. Exponential

$$e^{3+4i} = e^3\big(\cos(4) + i\sin(4)\big) = 20.0855(-0.6536 - 0.7568i)$$

$$= -13.1288 - 15.2008i$$

3. Complex Power

$$(3+4i)^{(1+i)} = \exp\big((1+i)\log_e(3+4j)\big)$$

$$= \exp((1+i) \times (1.6094 + i0.9273))$$

$$= \exp(0.6821 + 2.5367i) = -1.6271 + 1.1249i$$

4. Sine

$$\sin(3+4i) = \sin(3)\cosh(4) + i\cos(3)\sinh(4)$$

$$= 0.1411 \times 27.3082 + i\,(-0.9900) \times 27.2899 = 3.8532 - 27.0170i$$

5. Cosine

$$\cos(3+4i) = \cos(3)\cosh(4) - i\sin(3)\sinh(4)$$

$$= (-0.9900) \times 27.3082 + i0.1411 \times 27.2899 = 27.0351 - 3.8506i$$