Solving ODEs by Laplace Transforms

An ordinary differential equation (ODE) is an equation that relates a summation of a function $x(t)$ and its derivatives.

Laplace transforms provide a useful means of solving ordinary differential equations (ODEs).

Method

Given an ordinary differential equation in $x(t)$:

1. The first step is to take Laplace transforms of any occurrence of $x(t)$, its derivatives and and functions in the equation; typically we are saying that the Laplace transform of one side of the equation is equal to the Laplace transform of the other side.
2. Place the values for $x(t)$ and its derivatives when $t = 0$.
3. If we denote the Laplace transform of $x(t)$ as $X(s)$, then the next step is to make $X(s)$ the subject of the equation.
4. We then take the inverse Laplace transform of both sides of the equation to obtain an equation with subject $x(t)$; the solution.

Example 1

Find x when $\frac{dx}{dt} = t^2$ and x=2 when t=0.

Answer:

1. Take Laplace transforms of both sides:

$$L\left\{\frac{dx}{dt}\right\} = L\{t^2\}$$

Hence $sX(s) - x(0) = \frac{\mathbf{2}}{\boldsymbol{s}^{\mathbf{3}}}$, using Laplace transform 7

2. Hence $sX(s) - 2 = \frac{\mathbf{2}}{\boldsymbol{s}^{\mathbf{3}}}$

3. Hence $X(s) = \frac{2}{s} + \frac{2}{s^4}$

4. Hence $x(t) = 2 + \frac{1}{3}t^3$ using Laplace transforms 2 and 7.

Example 2

Find x when $\frac{d^2x}{dt^2} - 7\frac{dx}{dt} + 12x = 2$ and $x = 1$ and $\frac{dx}{dt} = 5$ when $t = 0$.

Answer:

1. Take Laplace transforms of both sides:

$$L\left\{\frac{d^2x}{dt^2}\right\} - 7L\left\{\frac{dx}{dt}\right\} + 12L\{x\} = L\{2\}$$

Hence $s^2X(s) - sx(0) - x'(0) - 7\big(sX(s) - x(0)\big) + 12X(s) = \frac{2}{s}$,

2. Hence $s^2X(s) - s - 5 - 7(sX(s) - 1) + 12X(s) = \frac{2}{s}$

substituting the initial conditions

3. Hence $(s^2 - 7s + 12)X(s) - s + 2 = \frac{2}{s}$

Hence $X(s) = \frac{s^2-2s+2}{s(s-3)(s-4)}$

Hence $X(s) = \frac{1}{6s} - \frac{5}{3(s-3)} + \frac{5}{2(s-4)}$, using partial fractions

Hence $x(t) = \frac{1}{6} - \frac{5}{3}e^{3t} + \frac{5}{2}e^{4t}$ using Laplace transforms 2 and 3.