Partial Differentiation

Partial differentiation has meaning in the context of the differentiation of functions of several variables. For example let $f(x,y,z)$ be a function of x,y,z. The first order partial derivative of f with respect to x is defined as

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y,z)-f(x,y,z)}{h},$$

whenever the limit exists. The partia
l derivatives of f with respect to y and z are defined similarly. The partial derivatives

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ and } \frac{\partial f}{\partial z}$$

and can be written f_x, f_y, f_z.

$\frac{\partial f}{\partial x}$ is determined by presuming that y and z are constant and then differentiating[1] in the normal way.

For example let $f(x,y,z) = x^2yz^3$ then $\frac{\partial f}{\partial x} = 2xyz^3, \frac{\partial f}{dy} = x^2z^3, \frac{\partial f}{\partial z} = 3x^2yz^2$.

By differentiating these functions again with respect to x, y, or z we obtain the higher order partial derivatives such as

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial y \partial z}.$$

For the example above with $f(x,y,z) = x^2yz^3$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2yz^3,$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2xz^3,$$

[1] Differentiation

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = 6xyz^2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2xz^3,$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 0,$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = 3x^2z^2,$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = 6xyz^2,$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = 3x^2z^2,$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = 6x^2yz.$$

Note that we find that changing the order of differentiation does not change the result.

For example

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$