

## Ordinary Differential Equations

Often in science and engineering we find that physical systems are such that the property to be studied is governed by an equation involving the derivatives<sup>1</sup> (often of different orders) of the (one) property of interest. The property of interest can be described as a function of one variable, in practice often a spatial or time variable. Such an equation is termed an *ordinary differential equation*(ODE). From the mathematical point of view the aim is to solve the differential equation for the function that represents the property of interest.

### Order

The highest order of the derivatives in the ODE is termed its *order*. The following are examples of first order ODEs:

1.

$$\frac{dy}{dx} - x = 0$$

2.

$$\frac{dy}{dx} + y = 0$$

The following are examples of second order ODEs

3.

$$\frac{d^2y}{dx^2} = y$$

4.

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 2$$

### General solution

Often, special mathematical or numerical methods are required to solve an ODE. One popular method is by using Laplace Transforms<sup>2</sup>.

Occasionally a solution of the differential equation can be found by inspection. For example  $y = \frac{1}{2}x^2$  can be deduced as a solution to example 1. However, we also note that adding any constant to this solution gives us another solution; we may express the *general* solution as  $y = \frac{1}{2}x^2 + C$ , a differential equation has a family of solutions.

Considering the second order ODE example 3, we could ask *which function do we differentiate twice with the result that the original function is returned?* Clearly  $e^x$  is a solution, but then so is any multiple of it;  $Ae^x$ . It can also be noted that  $e^{-x}$ , or any multiple of it is also a solution. In fact the general solution of equation 3 is

---

<sup>1</sup> [Differentiation](#)

<sup>2</sup> [Solving ODEs by Laplace Transforms](#)

$$y = Ae^x + Be^{-x} .$$

Note that the general solution of the first order ODE has one constant that can take any value and the general solution of the second order ODE has two constants which can take any value. And in general we find that an  $n^{\text{th}}$  order ODE has  $n$  constants that can take any value.

### Particular Solution

If we place *condition(s)* on the solution of an ODE then we can pinpoint a particular solution from the family of solutions to the ODE. A first order ODE has just one unknown constant, so only one further condition is required to determine a particular solution.

For example 1,  $\frac{dy}{dx} - x = 0$  with condition  $y(0) = 2$ .

Now the general solution is  $y = \frac{1}{2}x^2 + C$ , but the condition gives  $2 = \frac{1}{2}0^2 + C$  and hence that  $C = 2$ .

Hence the particular solution is  $y = \frac{1}{2}x^2 + 2$ .

For example 3,  $\frac{d^2y}{dx^2} = y$  with conditions  $y(0)=2$ , and  $\frac{dy}{dx}(0) = 0$

Now the general solution is  $y = Ae^x + Be^{-x}$ , but the first condition gives  $2 = Ae^0 + Be^{-0} = A + B$ . Hence  $A + B = 2$ .

If we differentiate the general solution we obtain the following:  $\frac{dy}{dx} = Ae^x - Be^{-x}$ . The second condition gives  $0 = A - B$ . Hence  $A = B$ .

The two equations in  $A$  and  $B$  have solution  $A=1$  and  $B=1$ .

Hence the particular solution is  $y = e^x + e^{-x}$ .

### Showing that a function is a solution of an ODE

If we are given a solution and we have to 'show' that it is a solution then the easiest way to do this is to substitute the solution and all its derivatives back into the ODE and confirm that it is a solution and show that it also satisfies the condition(s).

For example 3, if we had to show that  $y = e^x + e^{-x}$  is a solution of  $\frac{d^2y}{dx^2} = y$  with conditions  $y(0)=2$ , and  $\frac{dy}{dx}(0) = 0$  then first we would differentiate  $y$  twice and place the resulting functions in the ODE.  $\frac{dy}{dx} = e^x - e^{-x}$ ,  $\frac{d^2y}{dx^2} = e^x + e^{-x}$ ; confirming that  $y$  is a solution of the ODE. When  $x=0$  we also note that  $y = e^0 + e^{-0} = 2$ , and  $\frac{dy}{dx} = e^0 - e^{-0} = 0$ , thus confirming the conditions.