Matrix Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are the most fundamental characteristics of a square matrix[1]. For a square matrix A, the eigenvectors is the set of non-trivial (ie non-zero) vectors $\underline{x}$ that are simply scaled when they are multiplied by A, with the scalings being equal to the eigenvalues λ. That is the eigenvalues and eigenvectors of a matrix A are the non-trivial vectors $\underline{x}$ and scalars λ that satisfy the following equation:

$$A\,\underline{x} = \lambda\,\underline{x}.$$

Since $\lambda\,\underline{x} = \lambda\,I\,\underline{x}$, where I is the identity matrix then the equation can also be written in the alternative form

$$A\,\underline{x} - \lambda\,I\,\underline{x} = \underline{0}\,,$$

or

$$(A - \lambda\,I\,)\,\underline{x} = \underline{0}\,.$$

To find the eigenvalues, one method is to find the solutions λ to the equation $|A - \lambda\,I\,|=0$. Where $|\quad|$ represents the determinant[2]. For a 2x2 matrix this usually involves the solution of a quadratic equation[3].

<u>Example of a 2×2 matrix</u>

Find the eigenvalues of the matrix $A = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$.

To find the eigenvalues, we first obtain the solutions of $|A - \lambda\,I\,|=0$.

That is $\left|\begin{pmatrix} 3-\lambda & -2 \\ -1 & 4-\lambda \end{pmatrix}\right| = 0$, or $(3-\lambda)(4-\lambda) - (-2)(-1) = 0$.

Multiplying out the brackets gives $12 - 3\lambda - 4\lambda + \lambda^2 - 2 = 0$.

Tidying up, we obtain the quadratic equation $\lambda^2 - 7\lambda + 10 = 0$.

By factorising, it is found that $(\lambda - 2)(\lambda - 5) = 0$ and the eigenvalues are 2 and 5.

For each eigenvalue λ the corresponding eigenvector is found by finding a suitable $\underline{x}$ that satisfies the matrix-vector equation:

$$(A - \lambda\,I\,)\,\underline{x} = \underline{0}\,.$$

Note that there is no unique solution to this equation; if $\underline{x}$ is a solution then so is any multiple of $\underline{x}$. Often it is sufficient to state any one of these solutions; it is the ratio between the components of the eigenvector that matters. If a particular solution is to be determined then it is often the solution with a unit norm or the normalised eigenvector.

[1] Matrix Definitions
[2] Inverse of a 2x2 Matrix
[3] Solution of Quadratic Equations

Example of a 2×2 matrix

Find the eigenvectors of the matrix $A = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$.

It was found that the matrix has eigenvalues 2 and 5.

For the eigenvalue $\lambda = 2$, we need to solve $\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence the corresponding eigenvector has the form $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

For the eigenvalue $\lambda = 5$, we need to solve $\begin{pmatrix} -2 & -2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Hence the corresponding eigenvector has the form $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Example of a 3×3 matrix

Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$.

To find the eigenvalues, we first obtain the solutions of $|A - \lambda I| = 0$.

That is $\begin{vmatrix} 1-\lambda & -1 & 0 \\ 1 & 2-\lambda & 1 \\ -2 & 1 & -1-\lambda \end{vmatrix} = 0$, or

$$(1-\lambda)\big((2-\lambda)(-1-\lambda) - 1\big) - -1(1 \times (-1-\lambda) - -2) = 0.$$

Hence

$(1-\lambda)(-2 - 2\lambda + \lambda + \lambda^2 - 1) + (1-\lambda) = 0.$

Tidying up, we obtain the equation $(1-\lambda)(\lambda^2 - \lambda - 3) + (1-\lambda) = 0.$

Noticing the common factor of $(1-\lambda)$, the equation becomes

$$(1-\lambda)(\lambda^2 - \lambda - 2) = 0\,.$$

By factorising the quadratic expression, it is found that

$$(1-\lambda)(\lambda+1)(\lambda-2) = 0$$

and the eigenvalues are 1, -1 and 2.

Example of a 3×3 matrix

For $\lambda = 1$, $A - \lambda I = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -2 & 1 & -2 \end{pmatrix}$.

The corresponding eigenvector satisfies the equation $(A - \lambda I)\underline{x} = 0$. Hence for the eigenvector $\lambda = 1$, the eigenvector must satisfy the equation:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ -2 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first equation states that $-x_2 = 0$, hence $x_2 = 0$.

The second equation states that $x_1 + x_2 + x_3 = 0$ and so it follows that $x_1 = -x_3$.

The third equation duplicates the outcome of the second equation.

The eigenvector that this suggests is one of the form $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

For $\lambda = -1$, $A - \lambda I = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & 1 \\ -2 & 1 & 0 \end{pmatrix}$.

The corresponding eigenvector satisfies the equation $(A - \lambda I)\underline{x} = 0$. Hence for the eigenvector $\lambda = -1$, the eigenvector must satisfy the equation:

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & 1 \\ -2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first and third equation state that $2x_1 - x_2 = 0$. So that $x_2 = 2x_1$.

The second equation states that $x_1 + 3x_2 + x_3 = 0$ and so it follows that $x_3 = -7x_1$.

The eigenvector that this suggests is one of the form $\begin{pmatrix} 1 \\ 2 \\ -7 \end{pmatrix}$.

For $\lambda = 2$, $A - \lambda I = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & -3 \end{pmatrix}$.

The corresponding eigenvector satisfies the equation $(A - \lambda I)\underline{x} = 0$. Hence for the eigenvector $\lambda = 2$, the eigenvector must satisfy the equation:

$$\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first equation states that $-x_1 - x_2 = 0$, hence $x_2 = -x_1$.

The second equation states that $x_1 + x_3 = 0$ and so it follows that $x_3 = -x_1$.

The eigenvector that this suggests is one of the form $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

Hence the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$ has eigenvalues 1, -1 and 2 and corresponding

eigenvectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.