

Inverse of a 3x3 Matrix by elimination

A method for finding the inverse of a 3×3 matrix is described in this document.

The matrix $\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ will be used to illustrate the method.

In this method, first the original matrix is augmented with the identity matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array} \right).$$

We then work to turn the first matrix into the identity matrix by applying row operations. By applying the same operations to the other matrix, it will eventually become the inverse.

[Note. It is possible that a divide-by-zero may occur. This problem can be overcome by swapping rows. If this can't work then the matrix has no inverse and it is said to be singular.]

Eliminate elements (2,1) and (3,1)

Subtracting the top row from the 2nd and 3rd rows in turn eliminated the (2,1) and (3,1) elements. For the example:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array} \right).$$

Eliminate element (3,2)

In the example it is already zero.

Eliminate elements (1,3) and (2,3)

By subtracting $-2 \times$ the bottom row from the first row and $1 \times$ the bottom row from the second row, the (1,3) and (2,3) elements are eliminated:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 0 & 2 \\ 0 & -2 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array} \right).$$

Eliminate element (1,2)

Subtracting $-2 \times$ the middle row from the top row gives the following:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & -2 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array} \right).$$

Divide by diagonal elements

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -0.5 & 0.5 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array} \right).$$

The matrix on the right is now the identity matrix, so the matrix on the left is the inverse:

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & -0.5 & 0.5 \\ 1 & 0 & -1 \end{pmatrix}.$$

Spreadsheet solution

The accompanying spreadsheet finds the inverse by elimination.

Finding the inverse of a 3x3 matrix				www.mathematics.me.uk
Matrix place in yellow area	$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$	Inverse=	$\begin{pmatrix} -1 & 1 & 1 \\ 0 & -0.5 & 0.5 \\ 1 & 0 & -1 \end{pmatrix}$	
	$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$		matrix and identity matrix
1 1	$\begin{pmatrix} 1 & 2 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$		eliminate (2,1), (3,1)
0	$\begin{pmatrix} 1 & 2 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$		eliminate (3,2)
-2 1	$\begin{pmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$		eliminate (1,3),(2,3)
-1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$		eliminate (1,2)
1 -2 -1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 1 \\ 0 & -0.5 & 0.5 \\ 1 & 0 & -1 \end{pmatrix}$		divide by diagonals identity and inverse matrices result