Generalised Eigenvalue Problem

The standard eigenvalue problem[1] consists of finding the eigenvalues and eigenvectors of a matrix A that are the non-trivial vectors $\underline{x}$ and scalars λ that satisfy the following equation:

$$A\,\underline{x} = \lambda\,\underline{x}.$$

Eigenvalue problems can be posed in more general form. The most common form of generalised eigenvalue problem is the solution to a matrix-vector equation of the form:

$$A\,\underline{x} = \lambda\,B\,\underline{x}.$$

This can also be written

$$(A - \lambda B)\,\underline{x} = \underline{0}\,.$$

To find the eigenvalues, one method is to find the solutions λ to the equation $|A - \lambda I| = 0$. Where | | represents the determinant[2]. For a 2x2 matrix this usually involves the solution of a quadratic equation[3].

<u>Example of a 2×2 matrices</u>

Find the eigenvalues of the generalised eigenvalue problem

$$\begin{pmatrix} 0 & 2 \\ 4 & 2 \end{pmatrix} \underline{x} = \lambda \begin{pmatrix} -2 & 3 \\ 0 & 5 \end{pmatrix} \underline{x}.$$

To find the eigenvalues, we first obtain the solutions of $\left| \begin{pmatrix} 0 & 2 \\ 4 & 2 \end{pmatrix} - \lambda \begin{pmatrix} -2 & 3 \\ 0 & 5 \end{pmatrix} \right| = 0$.

That is $\left| \begin{pmatrix} 0 + 2\lambda & 2 - 3\lambda \\ 4 & 2 - 5\lambda \end{pmatrix} \right| = 0$, or $2\lambda(2 - 5\lambda) - 4(2 - 3\lambda) = 0$.

Multiplying out the brackets gives $4\lambda - 10\lambda^2 - 8 + 12\lambda = 0$.

Tidying up, we obtain the quadratic equation $10\lambda^2 + 16\lambda - 8 = 0$.

The solution is given by $\lambda = \frac{-16 \pm \sqrt{16^2 - 4 \times 10 \times 8}}{2 \times 10} = 0.8 \pm 0.4i$.

[1] Matrix Eigenvalues and Eigenvectors
[2] Inverse of a 2x2 Matrix
[3] Solution of Quadratic Equations

Example of a 2×2 matrices

Find the eigenvectors of the generalised eigenvalue problem, given that the eigenvalues are $0.8 \pm 0.4i$:

$$\begin{pmatrix} 0 & 2 \\ 4 & 2 \end{pmatrix} \underline{x} = \lambda \begin{pmatrix} -2 & 3 \\ 0 & 5 \end{pmatrix} \underline{x}.$$

For $\lambda = 0.8 + 0.4i$,

that is $\begin{pmatrix} 0 + 2(0.8 + 0.4i) & 2 - 3(0.8 + 0.4i) \\ 4 & 2 - 5(0.8 + 0.4i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

that is $\begin{pmatrix} 1.6 + 0.8i & -0.4 - 1.2i) \\ 4 & -2 - 2i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, giving an eigenvector $\begin{pmatrix} 2 + 2i \\ 4 \end{pmatrix}$.

For $\lambda = 0.8 - 0.4i$,

that is $\begin{pmatrix} 0 + 2(0.8 - 0.4i) & 2 - 3(0.8 - 0.4i) \\ 4 & 2 - 5(0.8 - 0.4i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

that is $\begin{pmatrix} 1.6 - 0.8i & -0.4 + 1.2i) \\ 4 & -2 + 2i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, giving an eigenvector $\begin{pmatrix} 2 - 2i \\ 4 \end{pmatrix}$.

For $\lambda = 2$, $A - \lambda I = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & -3 \end{pmatrix}$.

The corresponding eigenvector satisfies the equation $(A - \lambda I)\underline{x} = 0$. Hence for the eigenvector $\lambda = 2$, the eigenvector must satisfy the equation:

$$\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first equation states that $-x_1 - x_2 = 0$, hence $x_2 = -x_1$.

The second equation states that $x_1 + x_3 = 0$ and so it follows that $x_3 = -x_1$.

The eigenvector that this suggests is one of the form $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

Hence the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$ has eigenvalues 1, -1 and 2 and corresponding

eigenvectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ -7 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.